◐ 18 ◑
The Master Forest

There is only one road leading into the great Master Forest. When Craig reached the entrance, he saw an enormous sign hanging over the gate:

<u>THE MASTER FOREST</u>
ONLY THE ELITE ARE ALLOWED TO ENTER!

"Oh, heavens!" thought Craig. "I have no idea if they will let me in. I've never thought of myself as elite; in fact, I'm not quite sure I know what the word really means!"

At this point, an enormous sentinel blocked his way.

"Only the elite are allowed to enter!" he said in a terrible voice. "Are you one of the elite?"

"That depends on the definition of 'elite'," replied Craig. "How do you define an elite?"

"It's not how *I* define it that counts; it's how Griffin defines it."

"And who is Griffin?" asked Craig.

"Professor Charles Griffin—he is the resident bird sociologist of this forest, and he's boss around here. It's *his* definition that counts!"

"Then what is his definition?"

"Well," replied the sentinel in a softer tone, "his definition is a very liberal one. He defines an elite as anyone who wishes to enter. Do *you* wish to enter?"

"Of course!" said Craig.

"Then by definition, you're an elite and are free to enter.

I'm sure Professor Griffin will be delighted to meet you. His house is a mile and a half down the road. You can't miss it; it's in the shape of an enormous bird."

"That's a relief!" thought Craig as he wended his way to the house. "I wonder why Professor Griffin instituted such a strange rule, which in fact doesn't exclude *anybody*. What sort of a chap is this Griffin, anyway?"

Well, Craig was pleasantly surprised to find Professor Griffin a most kind and hospitable fellow. He was a gentleman in his mid-sixties with long flowing white hair and a long flowing white beard. He looked somewhat like the popular image of Moses, or of God the Father.

"Welcome!" said Griffin. "I hope you will find this forest of interest."

"I have come a long way," said Craig, "and I am very curious to know what birds you have here."

"A starling and a kestrel," replied Griffin.

"That's all?" asked Craig.

"And all birds derivable from them," replied Griffin.

"Oh, that's different! Are many birds derivable from just the starling and the kestrel?"

"Very many indeed!" replied Griffin, with a subtle and rather mysterious smile. "Are you familiar with the bluebird, the dove, the blackbird, the eagle, the bunting, the dickcissel, the becard, the dovekie, the bald eagle, the warbler, the cardinal, the identity bird, the thrush, the robin, the finch, the vireo, the queer bird, the quixotic bird, the quizzical bird, the quirky bird, the quacky bird, the mockingbird, the lark, the sage bird, the Turing bird, and the owl?"

"I know them all," replied Craig, "and you mean to tell me that *all* of them are derivable from *just* the two birds S and K?"

"Indeed they are!"

Craig sank back in thought.

"Perhaps that's not too surprising," Craig said at last. "I already know that all the birds you have just mentioned are derivable from the four birds B, T, M, and I, but I didn't know that those four birds were derivable from only two combinatorial birds. Those four birds are derivable from S and K?"

"Indeed they are," replied Griffin, "and many more birds you haven't ever heard of."

"Such as?"

"Now *this* should surprise you," replied Griffin. "From just S and K you can derive *any* combinatorial bird whatsoever! And there are infinitely many combinatorial birds!"

"Fantastic!" exclaimed Craig. "Only one thing puzzles me. How can this finite forest contain *infinitely* many birds?"

"Oh, they are not necessarily all here at the same *time*," replied Griffin. "This is an *evolving* forest, and there is an ancient legend explaining this.—Let's see, where did I put the book?"

Professor Griffin then rummaged around in his library-study and finally brought out one of the most worn books Craig had ever seen, although it showed signs of having originally had a most beautiful, if overly ornate, binding. The book was full of remarkable ancient paintings and drawings of birds—many of which were unfamiliar. It was written in an ancient script that Craig could not identify.

"Let me translate the legend as best I can," said Griffin. "I have some knowledge of the language, but not too much. As I understand it, it goes something like this:

"In the beginning, the forest gods started the forest with just two birds—the starling S and the kestrel K. There were already humans in the forest. New birds constantly came into existence in the following manner. A human would call out the name of some already existing bird y to some existing bird x; x would then respond by calling out the name of either some existing bird or of some nonexistent bird, but the marvelous thing is that if x named a nonexisting bird, the bird would then come into being! Thus new birds were constantly generated. The forest gods were wise in starting out with the starling and the kestrel, since from these two birds all combinatorial birds can be generated."

"That is the legend," continued Griffin. "Of course, it is only a legend, but it gives one food for thought. Some ornitheological historians have likened it to the story of Adam and Eve, though which of the birds S or K is Adam and which is Eve has been a matter of bitter controversy. Male historians like to think of S as Adam, but many female historians regard this as male chauvinism. More research is needed to settle the matter definitely. Ancient Chinese historians think of S as the yang, K as the yin, and their union as the all-embracing Tao. I could say much more about the legend's literary and historical aspects, but I'd like to get back to the purely scientific side of the story."

"The legend obviously has *some* foundation in truth," Griffin went on, "since it is really a fact that all combinatorial birds are derivable from just the two birds S and K."

"How is this known?" asked Craig.

"I will reveal the secret shortly," replied Griffin, "but first I would like you to try some concrete problems."

• 1 •

"Before we can get to first base," continued Griffin, "we must derive an identity bird I from S and K. Can you see how to do it?"

Inspector Craig fiddled around with this a bit, using pencil and paper, and got the solution. (The solution to this and the next three problems is incorporated into the text of "The Secret," later in this chapter.)

• 2 •

"Good!" said Griffin. "Now that we have the identity bird I, we are free to use it in future derivations, since it is derivable in terms of S and K. Most of our future derivations will be in terms of S, K, and I.

"Next, see if you can derive a mockingbird from S, K, and I—in fact, I'll give you a hint: A mockingbird can be derived from just S and I. Can you see how?"

Craig did not have too much difficulty with this one.

• 3 •

"Now let's take another familiar bird—the thrush," said Professor Griffin. "See if you can derive a thrush from S, K, and I."

Inspector Craig worked for quite a while on this one, but could not solve it.

"I'll give you some hints," said Griffin. "First find an expression X_1 satisfying the following two conditions:

1. X_1 is composed of just the letters S, K, I, and the variable x; the variable y should not be part of X_1.

2. The relation $X_1y = yx$ must hold, on the basis of the given conditions for S and K and also I."

Craig worked on this for a bit and found such an expression X_1.

"Now that I have it, what do I do with it?" asked Craig.

"The next step," replied Griffin, "is to find an expression X_2 having no variables at all—an expression built from just the letters S, K, and I—such that the relation $X_2x = X_1$ must hold. Then $X_2xy = X_1y$, and $X_1y = yx$ must hold, hence, the relation $X_2xy = yx$ must hold, so X_2 will be an expression for a thrush."

"Ah!" said Craig. "I begin to see light!"

• 4 •

"Now let's try a more complex one," suggested Griffin. "Try finding a bluebird in terms of S, K, and I. There are now three variables involved—x, y, and z. First find an expression X_1 in which z doesn't occur such that the relation $X_1z = x(yz)$

must hold. Then find an expression X_2 in which neither y nor z occurs—that is, x is to be the only variable—such that the relation $X_2y = X_1$ holds. Finally, find an expression X_3 in which no variable occurs such that the relation $X_3x = X_2$ must hold. Then X_3 will be an expression for a bluebird."

THE SECRET

Before telling the reader the general method of deriving any combinatorial bird from S and K, we will first solve the four problems given by Griffin.

First we derive an identity bird from S and K (Problem 1). Well, SKK is such a bird, because for any bird x, SKKx = Kx(Kx) = x.

We might remark that for *any* bird A, the bird SKA is an identity bird (why?), so, for example, SKS is also an identity bird. But for definiteness, we will take I to be the bird SKK, which is the usual convention.

Now let's derive a mockingbird from S, K, and I (Problem 2). As Professor Griffin remarked, we can get a mockingbird from just S and I. Well, since Ix = x, it follows that SIIx = Ix(Ix) = x(Ix) = xx, and so SII is a mockingbird.

Next for the thrush. This is trickier, since there are now two variables involved—x and y. As Professor Griffin suggested, let's first find an expression X_1 whose only variable is x such that the relation X_1y = yx must hold. Well, I is an expression such that Iy = y, and Kx is an expression such that Kxy = x, hence SI(Kx) is an expression such that SI(Kx)y = yx, because SI(Kx)y = Iy(Kxy) = Iyx = yx. So SI(Kx) is an expression in which y doesn't occur and which is such that SI(Kx)y = yx. We have thus found the desired expression X_1—namely, SI(Kx).

Now we follow Professor Griffin's second suggestion and look for an expression X_2 with no variables at all such that the relation $X_2x = SI(Kx)$ holds. Well, K(SI) is an expression such that $K(SI)x = SI$, and K is obviously an expression such that $Kx = Kx$, and so S(K(SI))K is an expression such that $S(K(SI))Kx = SI(Kx)$. We can check this: $S(K(SI))Kx = K(SI)x(Kx) = SI(Kx)$, since $K(SI)x = SI$. Therefore S(K(SI)) is a thrush. The reader can check this by computing S(K(SI))xy; he will end up with yx.

Finally, for the bluebird (Problem 4): We must first find an expression X_1 whose only variables are x and y such that the relation $X_1z = x(yz)$ holds. Well, since Kx is an expression such that $Kxz = x$ and y is an expression such that $yz = yz$, then S(Kx)y is an expression such that $S(Kx)yz = x(yz)$. *Check*: $S(Kx)yz = Kxz\,(yz) = x(yz)$. So X_1 can be taken to be S(Kx)y. It involves only the variables x and y.

Next we need an expression X_2 whose only variable is x such that the relation $X_2y = S(Kx)y$ holds. Here we have unexpected luck, since we can take X_2 to be S(Kx). *Note*: The expression S(Kx)y is already in the form X_2y, if y is not a variable of X_2. We are not often that lucky!

Finally, we need an expression X_3 with no variables at all such that the relation $X_3x = S(Kx)$ holds. Well, since KS is an expression such that $KSx = S$ and K is an expression such that $Kx = Kx$, then S(KS)K is the expression X_3 that we seek. *Check*: $S(KS)Kx = KSx(Kx) = S(Kx)$. Therefore S(KS)K must be a bluebird, as the reader can check by showing that $S(KS)Kxyz = x(yz)$.

We have by now pretty well illustrated the general method, which is this: Our expressions are built from the letters S, K, I, and variables x, y, z, w, v, and any others we might need. Let α stand for any one of the variables. For any expression X, call an expression X_1 an α-eliminate of X if the following two conditions hold:

1. The variable α does not occur in X_1.
2. The relation $X_1\alpha = X$ must hold. By this I do not mean that $X_1\alpha$ *is* necessarily the expression X, but only that the

equation $X_1\alpha = X$ is derivable from the defining conditions of S and K. For example, "KKα" and "K" are different expressions, but the relation $KK\alpha = K$ does hold, by virtue of the defining condition of the kestrel—namely that for *any* x and y, $Kxy = x$.

The fundamental problem, then, is this: Given an expression X and a variable α, how do we find an α-eliminate of X? This can always be done by a finite number of applications of the following four principles:

Principle 1: If X consists of just the variable α standing alone, then I is an α-eliminate of X. Stated otherwise, I is an α-eliminate of α. *Reason:* The variable α is obviously not part of the expression I and $I\alpha = \alpha$ does hold. Therefore I satisfies both the conditions of being an α-eliminate of α.

Principle 2: If X is an expression in which the variable α doesn't even occur, then KX is an α-eliminate of X. The reason is obvious: Since α doesn't occur in X, then it doesn't occur in KX and the relation $KX\alpha = X$ holds.

Principle 3: If X is a composite expression $Y\alpha$ and α doesn't occur in Y, then Y itself is an α-eliminate of X. Stated otherwise, if α doesn't occur in Y, then Y is an α-eliminate of $Y\alpha$. The reasons are obvious. As an example, yz is an x-eliminate of yzx, since x doesn't occur in yz and yz *is* an expression E such that $Ex = yzx$. Also KyI is an x-eliminate of KIyx, but KIy is *not* a y-eliminate of KIyx!

Principle 4: Suppose X is a composite expression YZ, and that Y_1 is an α-eliminate of Y, and Z_1 is an α-eliminate of Z. Then the expression of SY_1Z_1 is an α-eliminate of X. *Reason*: The relations $Y_1\alpha = Y$ and $Z_1\alpha = Z$ both hold, by hypothesis, and the relation $SY_1Z_1\alpha = Y_1\alpha(Z_1\alpha)$ holds, hence the relation $SY_1Z_1\alpha = YZ = X$ holds. Also α doesn't occur in Y_1 or in Z_1—by hypothesis that Y_1 and Z_1 are respectively α-eliminates of Y and Z—hence α doesn't occur in SY_1Z_1. Hence SY_1Z_1 is an expression X_1 in which α doesn't occur and which has the property that the relation $X_1\alpha = X$ must hold.

We note that Principle 4 reduces the problem of finding an α-eliminate of a complex expression YZ to the problem of finding α-eliminates of each of the shorter expressions Y and Z. To find one or both of these, you might have to use Principle 4 again, and perhaps again and again, but since the expressions involved are getting shorter and shorter, the process must finally terminate.

Let us consider some examples. Suppose we wish to find an x-eliminate of the expression yx(xy). In unabbreviated notation, the expression is (yx)(xy). We see that Principle 4 is the only one that is immediately applicable, and so we must first find an x-eliminate of yx and an x-eliminate of xy. Well, by Principle 3, y is an x-eliminate of yx. As to xy, we must again use Principle 4: Since I is an x-eliminate of x and Ky is an x-eliminate of y, then by Principle 4, SI(Ky) is an x-eliminate of xy. So y is an x-eliminate of yx and SI(Ky) is an x-eliminate of xy; therefore, by Principle 4, Sy(SI(Ky)) is an x-eliminate of yx(xy). The reader can check that Sy(SI(Ky))x = yx(xy).

On the other hand, suppose we wanted to find a y-eliminate of (yx)(xy). We must first find a y-eliminate of yx and a y-eliminate of xy. As to the former, since I is a y-eliminate of y and Kx is a y-eliminate of x, then SI(Kx) is a y-eliminate of yx. As to the latter, x is a y-eliminate of xy. So SI(Kx) is a y-eliminate of yx and x is a y-eliminate of xy, hence, by Principle 4, S(SI(Kx))x is a y-eliminate of yx(xy). The reader can check that the relation S(SI(Kx))xy = yx(xy) must hold.

Now that we know how to find an α-eliminate of X, for any variable α and any expression X, we can derive from S, K, and I any combinator to do any required job. If X has only one variable—say x—and we wish to find a combinator A such that the relation Ax = X holds, we take for A any x-eliminate of X. *Example*: Suppose we want a combinator A such that the relation Ax = x(xx) holds. Well, I is an x-eliminate of x, so SII is an x-eliminate of xx. Since I is an x-eliminate of x and SII is an x-eliminate of xx, then SI(SII) is an x-eliminate of x(xx). So a combinator A that works is SI(SII), as the reader can check.

Suppose we have an expression X involving two variables—say x and y—and we seek a combinator A such that the relation Axy = X holds. We first find a y-eliminate of X—call it X_1—and then we find an x-eliminate of X_1—call it X_2, so X_2 is the combinator we seek. As an example, suppose we want a combinator A such that for any x and y, Axy = yx(xy). Well, we have already found a y-eliminate of yx(xy)—namely S(SI(Kx))x. We must then find an x-eliminate of S(SI(Kx))x. We can arrange our work as follows:

1. K(SI) is an x-eliminate of SI.
2. K is an x-eliminate of Kx.
3. Therefore S(SI)K is an x-eliminate of SI(Kx).
4. KS is an x-eliminate of S.
5. Hence, according to steps 4 and 3 and Principle 4, S(S(KS)(S(SI)K))I is an x-eliminate of S(SI(Kx))x and is a
6. I is an x-eliminate of x.
7. Therefore, according to steps 5 and 6 and Principle 4, S(S(KS)(S(SI)K)) I is an x-eliminate of S(SI(Kx))x and is a combinator A doing the required job that Axy = yx(xy), as the reader can verify.

In short, if X is an expression in just two variables x and y, a combinator A that works for X—by which we mean that the relation Axy = X holds—is obtained by finding an x-eliminate of a y-eliminate of X—such an expression we call an x-y-eliminate of X. If X contains three variables x, y, and z, we find A by finding an x-eliminate of a y-eliminate of a z-eliminate of X—such an expression we call an x-y-z-eliminate of X. We have already done this for the expression x(yz), when we derived the bluebird.

Let us conclude with another example—finding a queer bird. Of course, we have already derived B and T from S and K, and in an earlier chapter we derived Q from B and T, but let us forget that we know this and see how to derive Q directly from S, K, and I.

The expression X is now y(xz). I will condense some of the steps. Thus, Ky is a z-eliminate of y; x is a z-eliminate of xz, so S(Ky)x is a z-eliminate of y(xz). Now we must find a y-eliminate of S(Ky)x. Well, S(KS)K is a y-eliminate of S(Ky)—I have condensed two steps—and Kx is a y-eliminate of x, so S(S(KS)K)(Kx) is a y-eliminate of S(Ky)x. Finally, we must find an x-eliminate of S(S(KS)K)(Kx). Well, an x-eliminate of S(S(KS)K) is K(S(S(KS)K)) and an x-eliminate of Kx is K, so S(K(S(S(KS)K))K) is an x-eliminate of S(S(KS)K)(Kx), and hence is a queer bird, as the reader can verify.

Of course, the procedure is applicable to an expression X with any number of variables. If X has four variables x, y, z, and w, we find the desired combinator by first finding the w-eliminate of X; then the z-eliminate of the result; then the y-eliminate of that result; and then the x-eliminate of *that*. Such an expression is called an x-y-z-w-eliminate of X. As an exercise, the reader should try to derive a dove D from S, K, and I. We recall that Dxyzw = xy(zw).

Some remarks are in order. First of all, the procedure we have described can be exceedingly tedious and often leads to much longer expressions than can be found by using cleverness and ingenuity. However, it is surefire, and is bound finally to result in the combinator you are seeking.

Secondly, it should be observed that the combinator you finally wind up with is not necessarily unique, because the method of finding α-eliminates can lead to several solutions, depending on the order in which you use the four principles. As an example, suppose we wish to find a z-eliminate of the expression xy. On the one hand, we can use Principle 2 and get K(xy) as a z-eliminate of xy. On the other hand, since Kx is a z-eliminate of x and Ky is a z-eliminate of y, then S(Kx)(Ky) is a z-eliminate of xy. Of course, the expression K(xy) is the simpler of the two, but K(xy) and S(Kx)(Ky) are both z-eliminates of xy, since K(xy)z = xy and also S(Kx)(Ky)z = Kxz(Kyz) = xy.

Another example: Suppose we want to find a y-eliminate of xy. On the one hand, x is a y-eliminate of xy, by Principle 3. On the other hand, since Kx is a y-eliminate of x by Principle 2 and I is a y-eliminate of y by Principle 1, then by Principle 4, S(Kx)I is also a y-eliminate of xy—a far more clumsy one than x, to be sure!

We see now that our process of finding an α-eliminate of an expression X is not deterministic; it can lead to more than one α-eliminate. It can be made completely deterministic by simply observing the following restriction: *Never use Principle 4 if any of the other three principles is applicable!*

With this restriction on the procedure, you can obtain only one α-eliminate of a given expression X. This deterministic procedure can be easily programmed on a computer, and those of you with home computers who like to work up software should find it an entertaining and profitable exercise to write a program to find combinators for any given expression.

◐ 19 ◑
Aristocratic Birds

Craig spent several weeks in the Master Forest and learned many interesting things from Professor Griffin.

"You seem to have known about many birds before you ever came here," remarked Griffin in one of their daily chats. "Where did you learn about them?"

"I learned about most of them from a certain Professor Adriano Bravura. Have you heard of him?"

"Oh, heavens!" cried Griffin. "He was my teacher! I spent several years in his forest. That's where I got my start."

"Several things have puzzled me," said Craig. "Professor Bravura showed me how to derive a large number of birds from just the four birds B, T, M, and I. Are *all* combinatorial birds derivable from these four birds?"

"Definitely not," replied Griffin. "The kestrel K cannot be derived from B, T, M, and I."

"Why is that?" asked Craig.

"This can be rigorously proved," replied Griffin. "The essential idea behind the proof is this:

"The bird K has what is known as a *cancellative* effect in that Kxy = x. Look at the right side of the equation Kxy = x. What has happened to the bird y? It has mysteriously disappeared—maybe it has flown away! Anyway, we say that y has been *canceled*, and hence that K has a *cancellative* effect. Likewise the bird K_2 obeying the condition $K_2xy = y$ is cancellative; the variable x has disappeared. I could name many more cancellative birds. Now, none of the birds B, M, T, and I are cancellative, and it is impossible to derive a cancellative bird from noncancellative birds. Therefore the cancellative bird K cannot be derived from B, T, M, and I."

"That's interesting," said Craig, "and that reminds me of another thing that has puzzled me. I once asked Professor Bravura whether there were any kestrels in his forest. He seemed somewhat upset by the question and replied in a strained voice: 'No! Kestrels are not allowed in this forest!' I felt like asking him why, but the subject obviously upset him. Do you know anything about this?"

"Oh, yes," said Griffin with a laugh. "You see, Bravura is somewhat of a purist and wants only aristocratic birds in his forest."

"Now, what on earth do you mean by an aristocratic bird?" asked Craig, in amazement.

"I got the term from Bravura," replied Griffin, still amused. "By an *aristocratic* bird, he simply means a combinatorial bird that is noncancellative."

"Why the term 'aristocratic'?" asked Craig.

"Well, you see, he is a bit eccentric in some of his ways. He comes from the ancient Italian nobility and has some rather old-fashioned aristocratic attitudes toward life. He regards any bird who 'cancels out' other birds as somehow lacking in nobility; he calls such birds *common* birds. The other birds he calls *aristocratic* birds.

"In a way, I can see his point," continued Griffin, "though of course I allow common birds in my forest, since they have a valuable mathematical function. And yet, if I have my choice of deriving an aristocratic bird either from S and K or from the four aristocratic birds B, T, M, and I, I tend to favor the latter. I am always a little uncomfortable using a common bird to derive an aristocratic one."

"Are all aristocratic birds derivable from B, T, M, and I?" asked Craig.

"Yes; there is a well-known recipe for deriving all aristocratic birds from B, T, M, and I—or more directly, from B, C, S, and I. I will show it to you sometime, if you like."

Note: The recipe is given in the appendix to this chapter.

"One thing strikes me as curious," said Craig. "From just two combinators—S and K—*all* combinators are derivable; yet we apparently need *four* aristocratic birds to derive all aristocratic birds. Why is this?"

"That's actually not true," replied Griffin. "It is true that of the four birds B, T, M, and I, no one of them is derivable from the other three. It is also true that of the four birds B, C, W, and I—which generate the same group as B, T, M, and I—none of them is derivable from the other three. It is also true that of the four birds B, C, S, and I—which again generate the same group as either of the other two foursomes—none of them is derivable from the other three. Nevertheless, there *does* exist a pair of aristocratic birds from which all aristocratic birds are derivable."

"That's interesting!" exclaimed Craig. "What are the two birds?"

"One of them is the identity bird I," replied Griffin, "and the other is a bird that may not be familiar to you—it is the *jaybird* discovered by J. Barkley Rosser in 1935. The bird J is defined by the following condition:

$$Jxyzw = xy(xwz)."$$

"That is a curious bird!" said Craig. "You are right; I'm not familiar with it. Please tell me more about it."

"Very well," said Griffin. "I will first show you that J is derivable from B, T, M, and I—more directly, from B, C, W, and I. In fact, J is derivable from just B, C, and W. Then I will show you that B, T, and M are derivable from J and I. It will then follow that the class of birds derivable from J and I is the same as the class of birds derivable from B, T, M, and I."

DERIVATION OF THE JAYBIRD

1 • Derivation of J

"There are several ways of deriving J from B, C, and W," said Griffin. "Perhaps the simplest uses the eagle E, the bird C*—that is, the cardinal once removed—and the bird C**—that is, the cardinal twice removed. Try to derive J from E, C*, C**, and W. Then express J in terms of B, C, and W."

GOING IN THE OTHER DIRECTION

"Now," said Griffin, "we proceed in the opposite direction. We start with the two birds J and I and set out to derive B, T, and M. We will have to rederive several familiar birds along the way, and the order will be very different from that of the original derivations from B and T. For example, we will have to derive C before B, and before *that* we must derive—of all birds!—the quixotic bird Q_1."

2 • Derivation of Q_1

"You recall the quixotic bird Q_1, defined by the condition $Q_1xyz = x(zy)$. Show that a quixotic bird is derivable from J and I."

3 • Derivation of the Thrush

"Next, derive a thrush T from Q_1 and I."

4 • Derivation of the Robin

"Next, from J and T, derive the robin R."

5 • Derivation of the Bluebird

"Now that we have R," said Griffin, "we can take C to be RRR, and so we have the cardinal. From C and Q_1 we can now get the bluebird B. Do you see how?

"Actually," added Griffin, "the bird C* is easily derivable from C and Q_1, and B is derivable from C* and Q_1. This may be the easiest path."

"Now we must derive the mockingbird," said Griffin. "This is the trickiest and most interesting part. It will be helpful first to derive a relative of J."

6 • The Bird J_1

"From the three birds J, B, and T, derive a bird J_1 satisfying the following condition:

$$J_1xyzw = yx(wxz)."$$

7 • The Mockingbird

"And now the mockingbird is derivable from C, T, and J_1. Show this.

"I'll give you a hint," added Griffin: "For any bird x, $J_1xTTT = xx$. You can easily verify this."

Griffin continued, "Now you see that the class of birds derivable from J and I is the same as the class of birds derivable from B, T, M, and I. If we started a bird forest with just the two birds J and I, we would ultimately get the same birds as if we had started with B, T, M, and I. The kestrel K would never appear, unless it flew in from another forest, nor would a whole host of birds derivable from S and K."

Note: The theory of combinators derivable from B, T, M, and I—or equivalently from B, C, W, and I, or from just J and I—is technically known as the λ I-calculus. The theory of combinators derivable from S and K is known as the λK-calculus. Neither theory can be said to be "better" than the other; each has applications which the other does not.

SOLUTIONS

1 · xy(xwz) = Exyxwz = C*Exxywz = W(C*E)xywz = C**(W(C*E))xyzw. And so we can take J to be C**(W(C*E)).

In terms of B, C, and W, we have J = B(BC)(W(BC(B(BBB)))).

2 · We originally took Q_1 to be BCB. But in terms of J and I, we can take Q_1 to be JI, because JIxyz = Ix(Izy) = x(Izy) = x(zy).

3 · We can take T to be Q_1I, because Q_1Ixy = I(yx) = yx.

In terms of J and I, we can take T to be JII.

4 · JTxyz = Tx(Tzy) = Tx(yz) = yzx. Therefore JT is a robin.

In terms of J and I, we can therefore take R to be J(JII).

5 · We can take C* to be Q_1C, because C(Q_1C)xyzw = Q_1Cyxzw = C(xy)zw = xywz.

Then we can take B to be C*Q_1, because C*Q_1xyz = Q_1xzy = x(yz). Therefore C*Q_1 is a bluebird.

In terms of C and Q_1, B = (Q_1C)Q_1.

6 · BJTxyzw = J(Tx)yzw = Txy(Txwz) = yx(wxz). And so we take J_1 to be BJT.

7 · First let's follow Griffin's hint. Well, J_1xTTT = Tx(TxT) = Tx(Tx) = (Tx)x = Txx = xx.

Now, J_1xTTT = CJ_1TxTT = C(CJ_1T)TxT = C(C(CJ_1T)T)Tx. And so we take M to be C(C(CJ_1T)T)T. In terms of B, C, T, and J, M = C(C(C(BJT)T)T)T. A bizarre expression for a mockingbird indeed! But it works.

As Curry remarked, the combinator J seems extremely artificial. It certainly does, but it yields the theoretically interesting result that the class of all combinators derivable from B, T, M, and I contains two combinators from which all the others are derivable.

APPENDIX

For the reader who is interested, here is a recipe for deriving any aristocratic bird from the four birds B, C, S, and I.

We use the notion of α-*eliminates* as defined in the last chapter. Let us call an expression X a *nice* expression if it is built up from the letters B, C, S, T, and variables. The letter K is definitely *not* allowed! What now needs to be shown is that if X is any nice expression, and if α is any variable *which actually occurs in X*, then we can find a *nice* α-*eliminate* of X. The procedure we describe for finding it will in fact lead to a *unique* nice α-eliminate of X, which we will call the *distinguished* α-eliminate of X. Our procedure is again a *recursive* one in the sense that the problem of finding the distinguished α-eliminate of a compound expression XY is sometimes reduced to the problem of first finding the distinguished α-eliminate of X and the distinguished α-eliminate of Y.

Here are the rules of the procedure.

Rule 1: The distinguished α-eliminate of α itself is I.

Rule 2: If α does not occur in X, then the distinguished α-eliminate of Xα is simply X.

Rule 3: Now consider a compound expression XY in which α occurs. Then α must occur in either X or Y and possibly in both. We assume that Y does not consist of just the variable α, because otherwise the situation reduces to Rule 2. Let X_1 be the distinguished α-eliminate of X and let Y_1 be the distinguished α-eliminate of Y. Assuming you have already found X_1 and Y_1, here is how to find the distinguished α-eliminate of XY.

a. If α occurs in both X and Y, then take SX_1Y_1 as the distinguished α-eliminate of XY.

b. If α occurs in Y but not in X, then take BXY_1 as the distinguished α-eliminate of XY.

c. If α occurs in X but not in Y, take CX_1Y as the distinguished α-eliminate of XY.

Let's consider some examples:

1. What is the distinguished z-eliminate of yz? By Rule 2 it is y.

2. What is the distinguished z-eliminate of zy? The applicable rule here is part c of Rule 3: We must first obtain the distinguished z-eliminate of z; this is I, by Rule 1. Then by part c of Rule 3, the distinguished z-eliminate of zy is CIy. We can check; CIyz = Izy = zy.

3. Find the distinguished y-eliminate of y(xy). *Solution*: I is the distinguished y-eliminate of y and x is the distinguished y-eliminate of xy, so SIx is the distinguished y-eliminate of y(xy)—according to part a of Rule 3.

4. Find the distinguished y-eliminate of z(xy). *Solution*: x is the distinguished y-eliminate of xy, so Bzx is the distinguished y-eliminate of z(xy). Checking this is obvious: Bzxy = z(xy).

Exercise: In terms of B, C, S, and I, find a combinator A satisfying the condition Axyz = xz(zy). The problem should be divided into three parts:

a. Find the distinguished z-eliminate of xz(zy).

b. Find the distinguished y-eliminate of the expression obtained in (a).

c. Find the distinguished x-eliminate of the expression found in (b). This is the desired expression A. Verify that it really works!

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Craig's Discovery

"I have a question for you," said Craig to Griffin the next day. "Consider the class of all birds derivable from just the three birds B, T, and I. Now—"

"Oh, that's an interesting class!" interrupted Griffin. "This class has been studied by J. B. Rosser in connection with certain logics in which duplicative birds like M, W, L, S, and J have no place. Rosser was therefore interested in the class of birds derivable from B, C, and I, but this is the same class as you have just described. Now, what do you want to know about it?"

"Can you replace the three birds B, T, and I by just *two* members of this class from which all members of the class are derivable?"

"That's an interesting question!" said Griffin. "I've never thought about it."

"Well," said Craig, "I was thinking about it last night, and I believe I've found the answer. I have discovered a bird derivable from just B and T such that both B and T are derivable from this bird together with I."

"How interesting!" cried Griffin. "What bird is that?"

"It is the goldfinch G," replied Craig. "The bird defined by the condition Gxyzw = xw(yz). Well, I have been able to derive B and T from G and I, hence the class of birds derivable from B, T, and I is the same as the class of birds derivable from G and I."

"That's neat!" said Griffin. "How do you get B and T from G and I?"

"Getting T is simple," replied Craig, "but I had a lot of trouble getting B. Here, let me show you what I have done.

• 1 •

"The first thing I did was to derive the *quirky* bird—the bird Q_3 satisfying the condition $Q_3xyz = z(xy)$. This bird is easily derivable from G and I. Do you see how?"

• 2 •

"Although this is a side issue, I might mention that the thrush T is easily derivable from Q_3 and I—hence from G and I. Can you see how?"

• 3 •

"The important thing is to get the cardinal C," said Craig. "Can you see how to get C from G and I?"

• 4 •

"Now that we have C," said Craig, "we have CC, which is a robin R. Then from R, G, and Q_3 we can get the queer bird Q. Do you see how? Then, of course, once we have Q and C, we have CQ, which is a bluebird B."

"Excellent!" said Griffin, after he solved these problems. "I'm delighted that after such a short time, you have begun to do original work in this field!"

SOLUTIONS

1 · GI is a quirky bird, since $GIxyz = Iz(xy) = z(xy)$. So we take Q_3 to be GI.

2 · Q_3I is a thrush, since $Q_3Ixy = y(Ix) = yx$.

3 · GGII is a cardinal, since GGIIxyz = Gx(II)yz = GxIyz = xz(Iy) = xzy.

4 · GRQ_3 is a queer bird, since GRQ_3xyz = $Ry(Q_3x)z$ = Q_3xzy = y(xz). So we take Q to be GRQ_3.

We might note that GR is in fact a cardinal once removed, as the reader can easily verify, and so we could take C* to be GR. Then C^*Q_3 is a queer bird Q.