55. "Difficult Crossing" Problems

As a continuation of the analysis of §32, where the Hitchcock-Koopmans transportation problem was considered, let us discuss a type of mathematical puzzle that appears frequently in books on mathematical recreations.

A typical poser is the following:

"A group consisting of three cannibals and three missionaries seeks to cross a river. A boat is available which will hold two people, and which can be navigated by any combination of cannibals and missionaries involving one or two people. If the missionaries on either side of the river, or in the boat, are outnumbered at any time by cannibals, the cannibals will indulge in their anthropophagic tendencies and do away with the missionaries. What schedule of crossings can be devised to permit the entire group of cannibals and missionaries to cross the river safely?"

In the next section we shall formulate the problem in more general terms and then resolve it by means of the functional equation technique.

56. General Problem

Let us now consider the more general situation in which we start with m_1 cannibals and n_1 missionaries on one side of the river and m_2 cannibals and n_2 missionaries on the other. Let the rule be that on one bank we have a constraint $R_1(m_1, n_1) \geq 0$ to prevent the missionaries from being devoured, a similar constraint $R_2(m_2, n_2) \geq 0$ on the other, and a constraint $R_3(m, n) \geq 0$ in the boat, capable of carrying at most k people.

Given the integers m_1, n_1, m_2, n_2, it is not at all clear when it is possible to schedule a safe crossing. Consequently, we shall begin by treating the following problem. Starting with the given initial data, what is the maximum number of people that can be transported from one bank, say bank one, to the other, without permitting cannibalism?

57. Functional Equations

Since the total number of cannibals and of missionaries stays constant throughout the process, the state of the system at any time is specified by the numbers m_1 and n_1 defined above.

Let us then introduce the function

(1) $f_N(m_1, n_1) =$ the maximum number of people on the second bank at the end of N stages, starting with m_1 cannibals and n_1 missionaries on the first bank and quantities m_2 and n_2 respectively on the second bank.

We shall suppose that it is permissible at any stage to send no people back to the first bank from the second bank if everybody is already on the second bank.

One stage of the process consists of sending x_1 cannibals and y_1 missionaries from the first bank to the second bank and then of sending x_2 cannibals and y_2 missionaries back to the first bank.

Using the principle of optimality, we obtain the recurrence relation

$$(2) \qquad f_N(m_1, n_1) = \max_{x,y} f_{N-1}(m_1 - x_1 + x_2, n_1 - y_1 + y_2),$$

for $N \geq 2$, where the variables x_1, x_2, y_1, y_2 are subject to the constraints

$$\begin{array}{lll}
(3) & (a) & 0 \leq x_1 \leq m_1, \quad 0 \leq y_1 \leq n_1, \\
 & (b) & 0 \leq x_2 \leq m_2 + x_1, \quad 0 \leq y_2 \leq n_2 + y_1, \\
 & (c) & x_1 + y_1 \leq k, \quad x_2 + y_2 \leq k, \\
 & (d) & R_3(x_1, y_1) \geq 0, \quad R_3(x_2, y_2) \geq 0, \\
 & (e) & R_1(m_1 - x_1, n_1 - y_1) \geq 0, \\
 & (f) & R_1(m_1 - x_1 + x_2, n_1 - y_1 + y_2) \geq 0, \\
 & (g) & R_2(m_2 + x_1, n_2 + y_1) \geq 0, \\
 & (h) & R_2(m_2 + x_1 - x_2, n_2 + y_1 - y_2) \geq 0.
\end{array}$$

There are sets of x_1, x_2, y_1, y_2 satisfying these constraints, since by assumption $x_1 = x_2 = y_1 = y_2 = 0$ satisfies them.

For $N = 1$, we have

$$(4) \qquad f_1(m_1, n_1) = \max_{x,y} [(m_2 + x_1) + (n_2 + y_1)],$$

where x_1, y_1 are subject to the foregoing constraints.

58. Discussion

For small values of N and of m_1, m_2, n_1, n_2, the values of $f_N(m_1, n_1)$ can readily be computed by hand. In many cases, the constraints will be of such restrictive type that there will be a unique feasible policy, which automatically will be the optimal policy.

In the foregoing manner we simultaneously determine the minimum number of crossings necessary for the transference of all the people from one side of the bank to the other, whenever this is possible. To obtain this minimum number, we continue the process until a value of N is obtained for which $f_N = m_1 + m_2 + n_1 + n_2$.

59. Numerical Solution

We illustrate the above algorithm by solving the problem stated in §55.

We first recognize that only certain initial states for the N stage process are possible. All others lead to immediate cannibalism. If we let (i, j) be the state of i missionaries and j cannibals being on the starting bank of the river and $3 - i$ missionaries and $3 - j$ cannibals on the second bank, then only the states (0, 1), (1, 1), (3, 1), (0, 2), (2, 2), (3, 2), (0, 3) and (3, 3) are possible. We use the algorithm of §57 to compute

$$f_1(0, 1) = 6,\ f_1(1, 1) = 6,\ f_1(3, 1) = 2,\ f_1(0, 2) = 6,$$
$$f_1(2, 2) = 3,\ f_1(3, 2) = 2,\ f_1(0, 3) = 4,\ f_1(3, 3) = 1.$$

Observing that if $f_k(i, j) = 6, f_{k+l}(i, j) = 6$ for $l = 1, 2, \ldots$, we iterate the recurrence relation for all non-six values and get

$$f_2(3, 1) = 3,\ f_2(2, 2) = 4, f_2(3, 2) = 2, f_2(0, 3) = 6, f_2(3, 3) = 2.$$

Continuing the process,

$$f_3(3, 1) = 4,\ f_3(2, 2) = 6,\ f_3(3, 2) = 3,\ f_3(3, 3) = 2,\ f_4(3, 1) = 6,$$
$$f_4(3, 2) = 4,\ f_4(3, 3) = 3, f_5(3, 2) = 6, f_5(3, 3) = 4, f_6(3, 3) = 6.$$

Therefore the required number of crossings, starting with 3 cannibals and 3 missionaries on bank one, is 6. The optimal policy is easily determined if the maximizing decision is recorded at each stage.

Comments and Bibliography

The problem of the maximization or minimization of a function of N variables is one of the major questions of analysis and, quite naturally, an enormous amount of effort has been devoted to it. Since we are interested only in functions of quite special form, we have paid no attention to any of the general techniques that exist. For a discussion of the classical method of "steepest descent," see

P. C. Rosenbloom, "The method of steepest descent," *Numerical Analysis, Proceedings of the Sixth Symposium in Applied Mathematics*, McGraw-Hill Book Co., Inc., New York, 1956.

Many references will be found there. See also

J. Todd, "Motivation for working in numerical analysis," *Comm. Pure Appl. Math.*, vol. 13, 1955, pp. 97–116.

§14. For rigorous discussions of the fundamental notion of convexity, see

T. Bonessen and W. Fenchel, *Theorie der Konvexen Korper*, Ergebnisse der Math., vol. 3, 1934.

H. G. Eggleston, *Convexity*, Cambridge Tracts No. 47, Cambridge University Press, Cambridge, 1958.

For some analytic applications of the techniques of this section, see

E. F. Beckenbach and R. Bellman, *Inequalities*, Ergebnisse der Math., Springer, Berlin, 1961.

We use the duality of Euclidean geometry to provide a further decomposition of processes. This property will be utilized again in the study of the calculus of variations.

§15. For the interpretation of the Lagrange multiplier as a "price," see the book by P. Samuelson referred to at the end of Chapter I, and Appendix II by S. Dreyfus and M. Freimer.